Network Theory

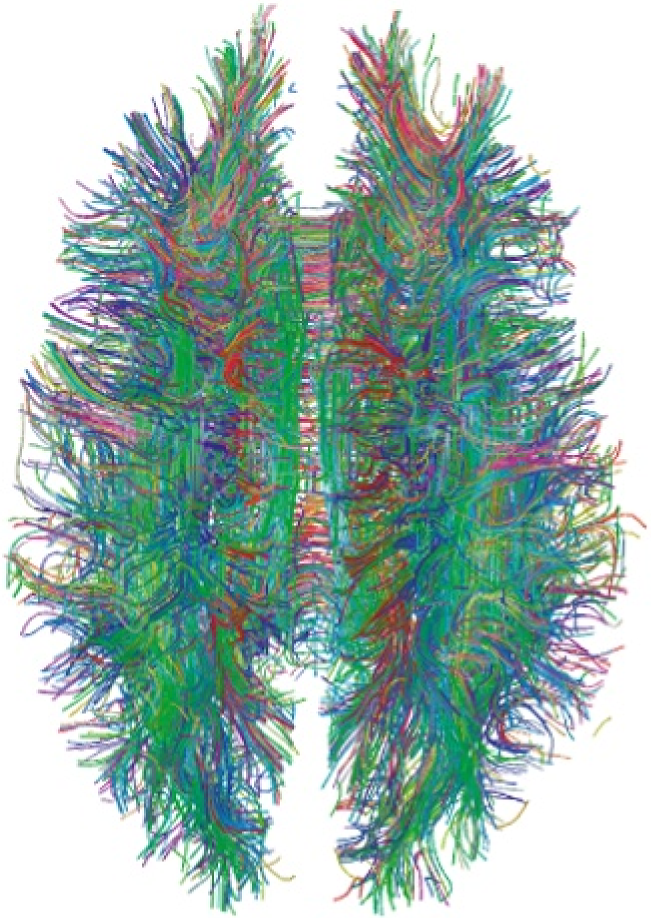
- Graph metrics for undirected binary graphs
  - ➢ degree, hubs, centrality
  - ➢ communities
  - ➢ scale-free and small-world graphs
- Directed and weighted graphs
- Application to synthetic and real datasets
  - ➢ Python library: networkx

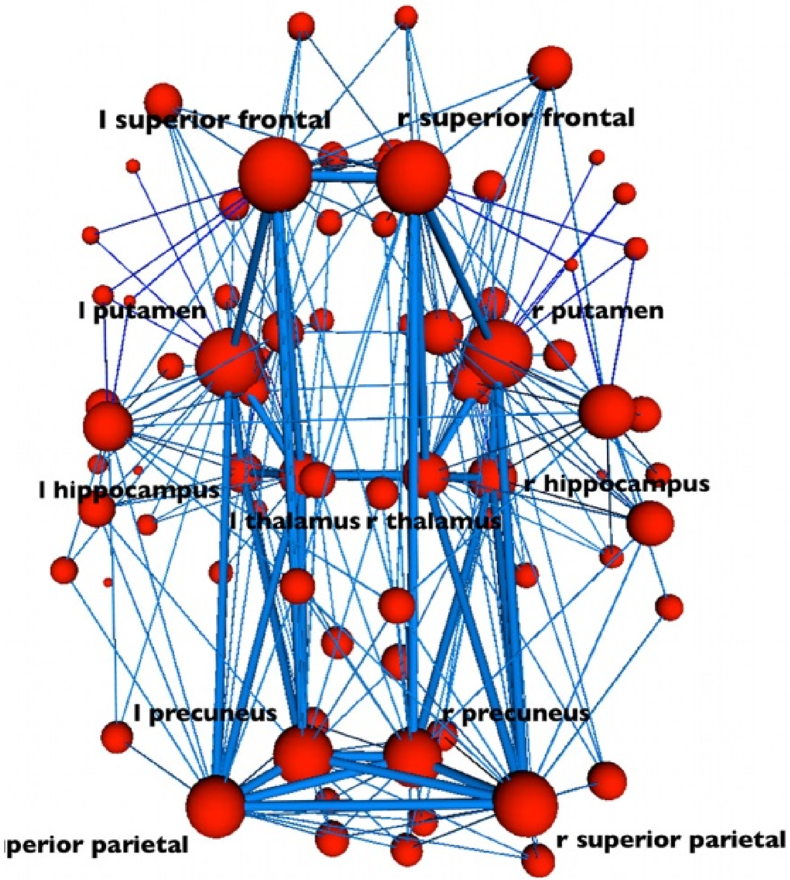
References

- Articles and reviews:
  - Krendl et al. (2022) *SCAN*: networks in social and cognitive neuroscience
  - Douw et al. (2023) *Netw Neurosci*: clinical perspectives
  - van den Heuvel and Sporns (2011) *J Neurosci*: rich-club in structural connectome

- To go further for application to fMRI dynamics:
  - Gilson et al. (2020) *Neuroimage*: model-based analysis of whole-brain dynamics based on fMRI data

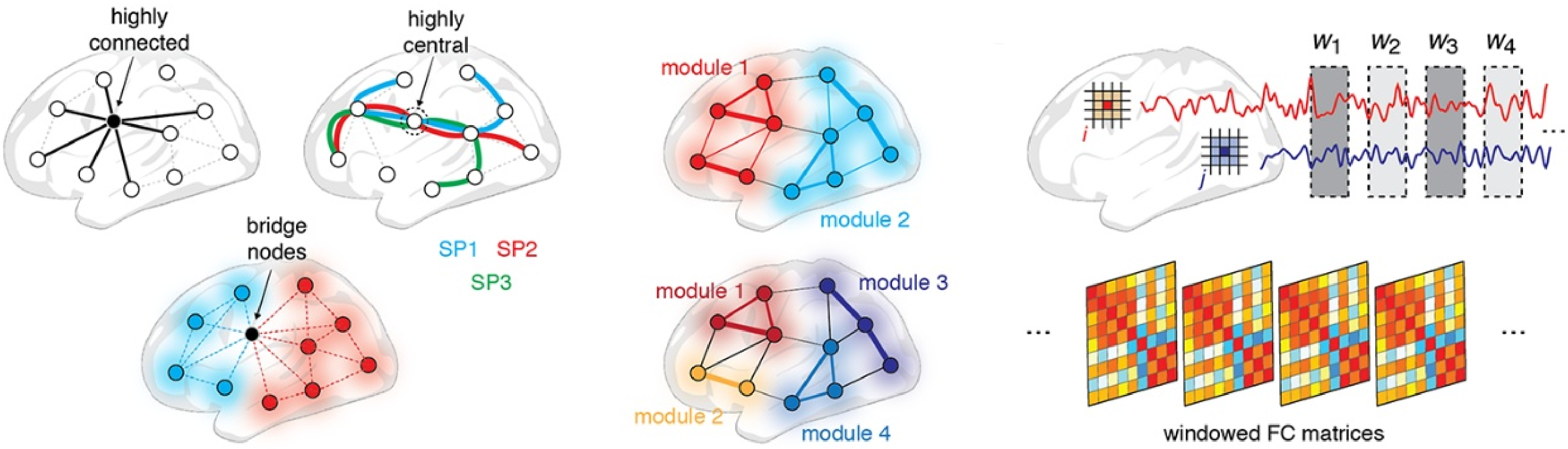
Rich-Club in Structural Connectome (White-Matter Fibers)





Hagman et al. (2008) *PLoS Biol*; van den Heuvel et al. (2011) *J Neurosci*

Network Analysis of Functional MRI



- Links reflect correlations of fMRI signals
- How to interpret network measures?

Lrendl et al. (2022) *SCAN*

Class 2: Network Theory

- **Graph metrics for undirected binary graphs**
  - **degree, hubs, centrality**
  - communities
  - scale-free and small-world graphs
- Directed and weighted graphs
- Application to synthetic and real datasets
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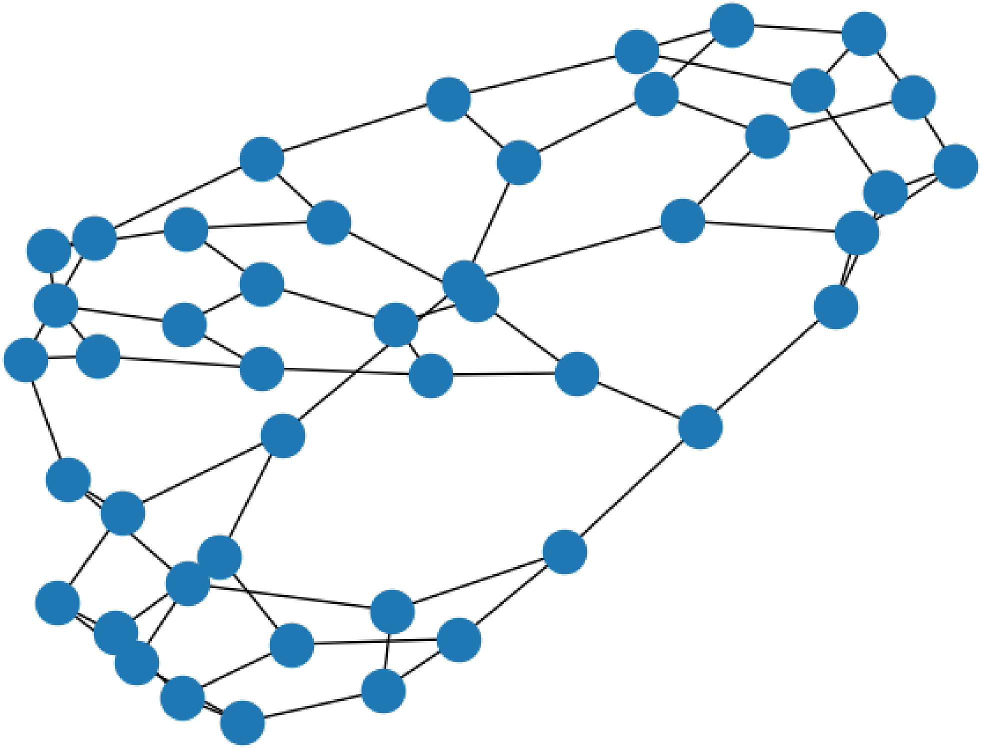
What is a Network/Graph?



- Nodes
- Edges

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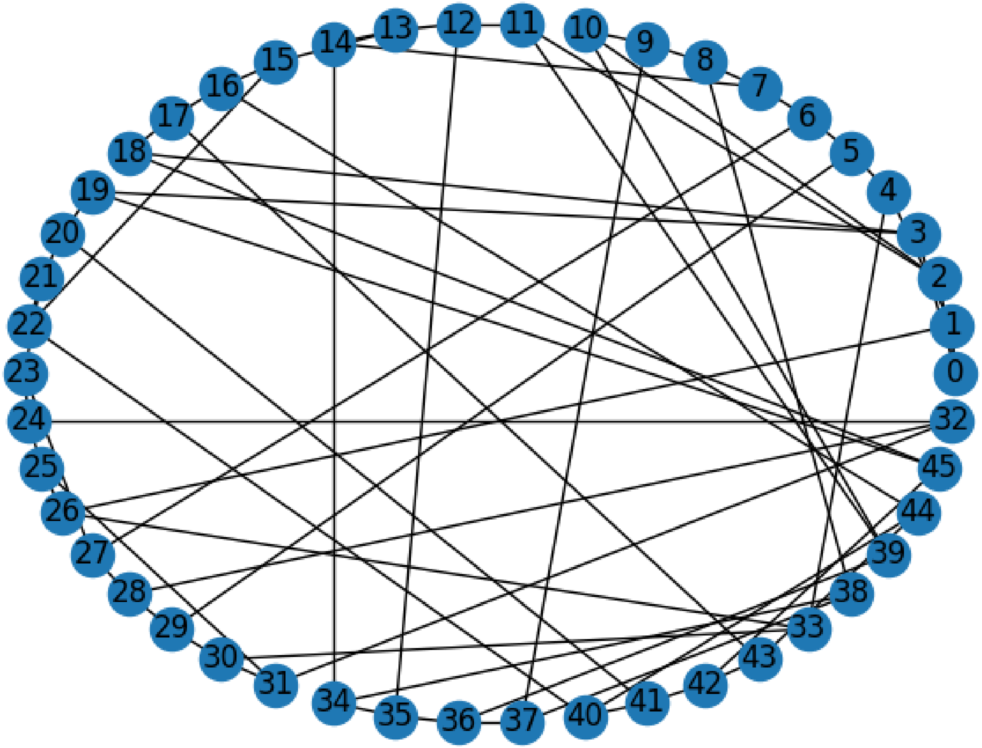

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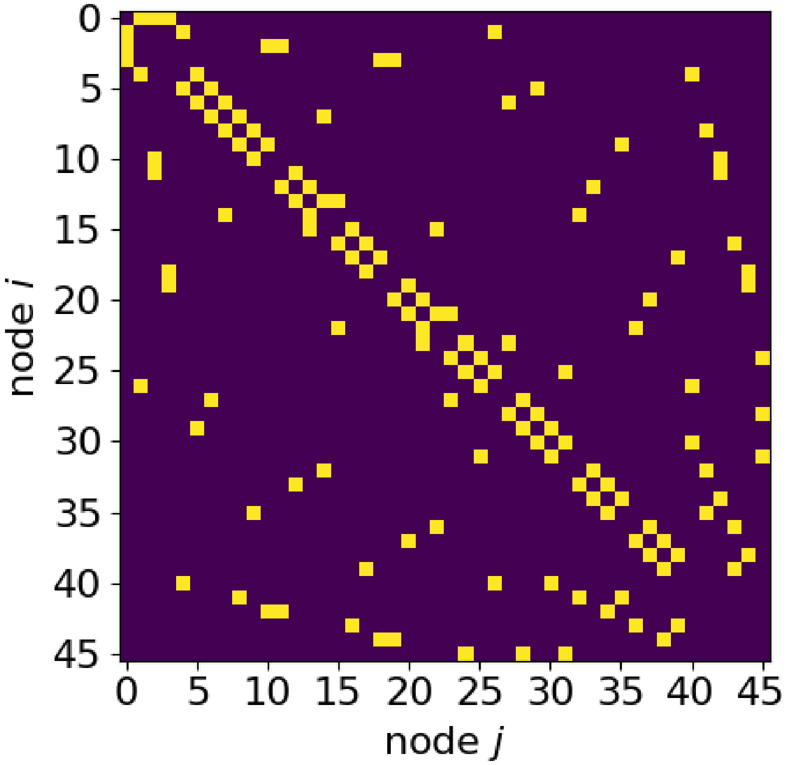
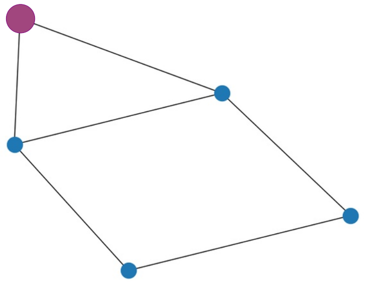


- Nodes *i*
- Edges *A_ij*
- Adjacency matrix *A*

How important is a node?

- Degree of a node: number of connections
  - a node with high degree is called a ‘hub’



degree = 2

How important is a node?

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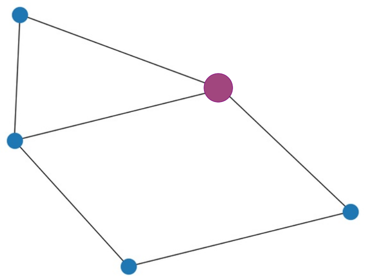
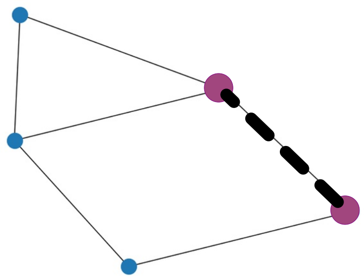


degree = 3

How important is a node?

- Degree of a node: number of connections
  - a node with high degree is called a ‘hub’
- Shortest path: for a given pair of nodes, the shortest path is the minimum number of links between them (if they are directly or indirectly connected)
  - the distribution of shortest paths shows how “easy” it is to “travel” in the network



shortest path length = 1

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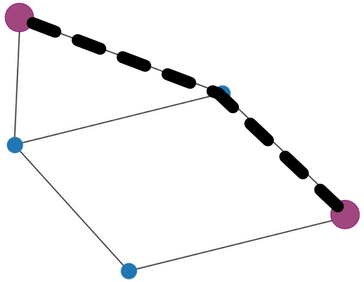

shortest path length = 2

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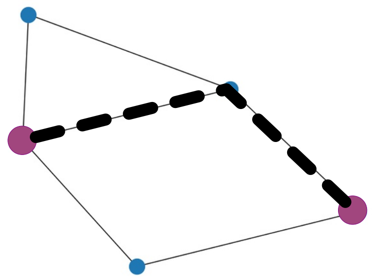

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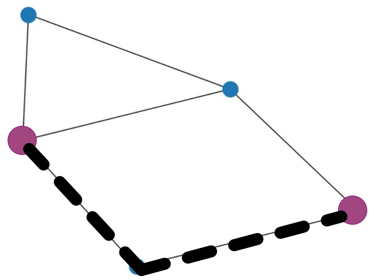
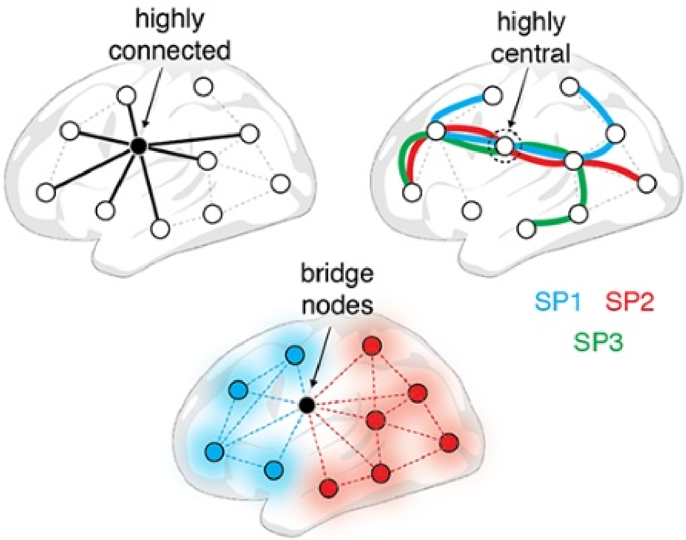


shortest path length = 2

How important is a node?

- Centrality: one concept, many flavors
  - degree centrality for a node is the fraction of neighbors (w.r.t. all remaining nodes)
  - betweenness centrality of a node is the sum of the fraction of all-pairs' shortest paths that pass through
  - eigenvector centrality comes from the spectral analysis of the adjacency matrix



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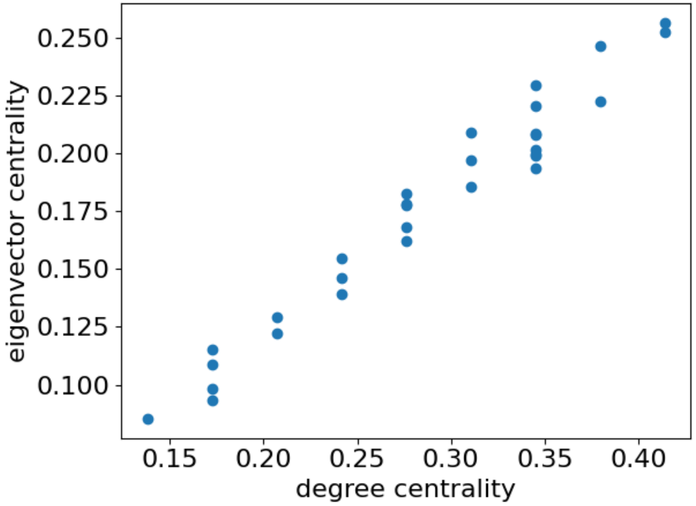

Mathematical Formulation based on Adjacency Matrix

- Adjacency matrix: A_{ij}
- Degree: $d_i = \sum_j A_{ij}$ so in matrix form: $d = A\, e$ with $e = (1, \ldots, 1)^T$

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- Number of paths from *j* to *i* of length *l* $(A^l)_{ij}$
- Dominating eigenvector and eigenvalue: $A v = \lambda v$
  - ➢ λ with largest real part among all eigenvalues
  - ➢ eigenvector centrality $|v_i|$

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  - **communities**
  - scale-free and small-world graphs
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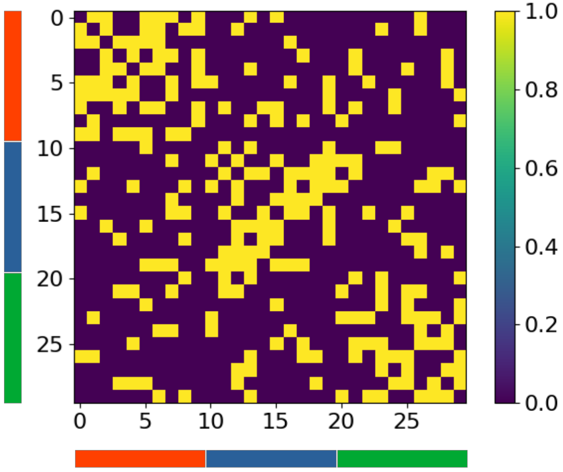
Communities

- Groups of nodes that are densely connected
- Detection using metrics that quantify within-group density compared to outside
- How likely are two nodes to be connected?



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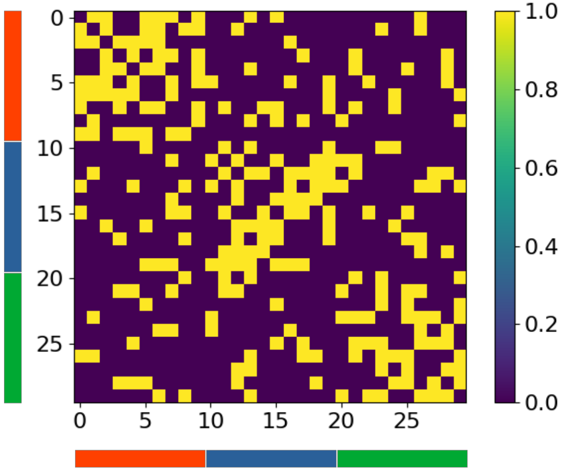
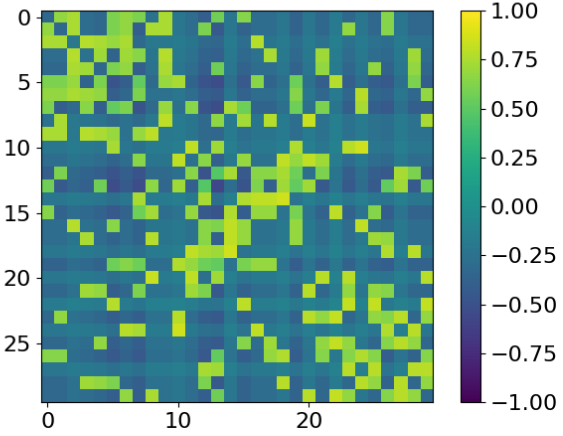


modularity matrix

$$A_{ij} - A_{ij}^{ref}$$



$$A_{ij}^{ref} = \frac{d_i d_j}{\sum_k d_k}$$ ref calculated from degree d_i

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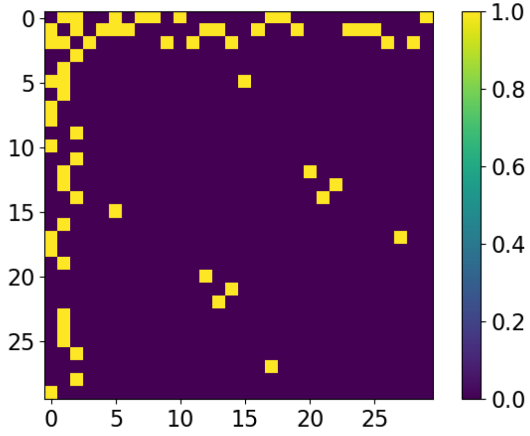

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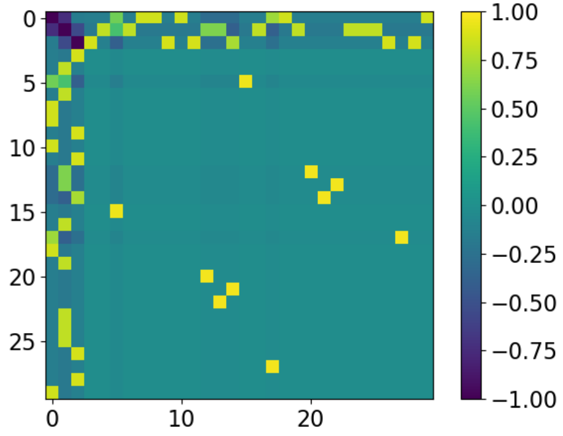

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ref calculated from degree d_i

Greedy Algorithm based on Modularity

- Modularity to assess quality of community grouping (C_1, C_2, ...)

$$Q=\sum_{i,j\in C_l} A_{ij}-\frac{d_i d_j}{\sum_k d_k}$$

connection probability
“by chance”

- Start with singletons of nodes $\{i\},\{j\},\ldots$
- Check nodes that are “strongly connected” $\Delta Q=A_{ij}-\frac{d_i d_j}{\sum_k d_k}$
- Iterative process to merge subgroups $\{i\},\{j\}\rightarrow\{i,j\}$
- Repeat so long as Q increases
  - list of communities: $\{0,2,5,\ldots\},\{1,3,4,\ldots\},\ldots$

Girvan & Newman (2002) *PNAS*, Newman (2006) *Phys Rev E*

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  - **scale-free and small-world graphs**

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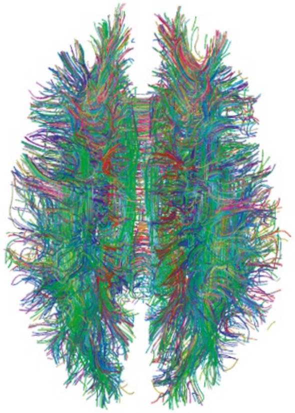
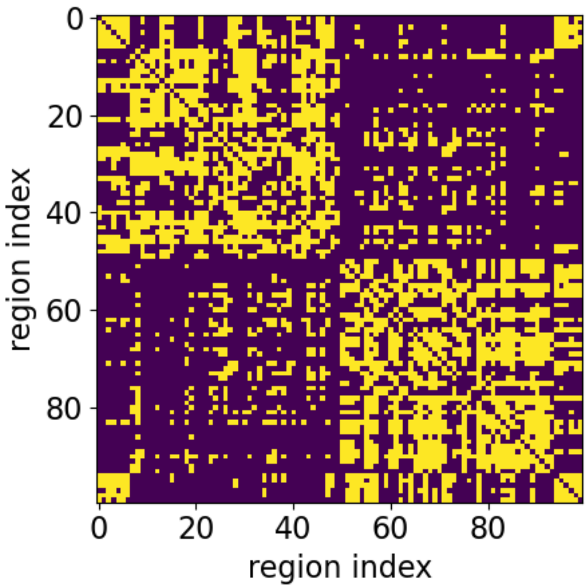
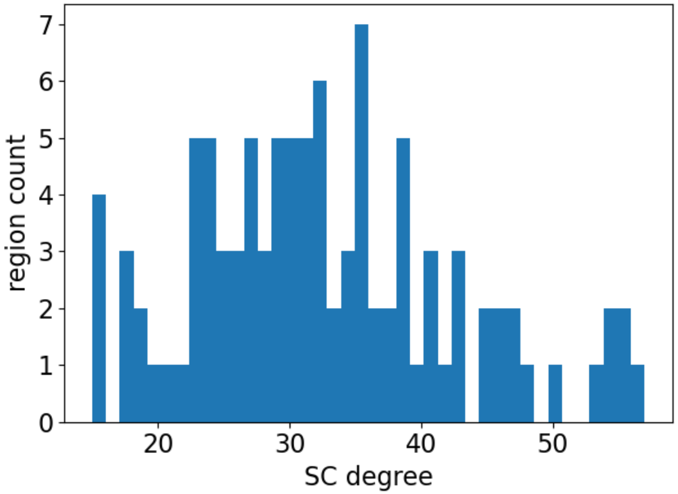
Synthetic Models for Real Data

- Connectivity in real data? Example of structural connectome
- Distribution of degree across cortical regions?

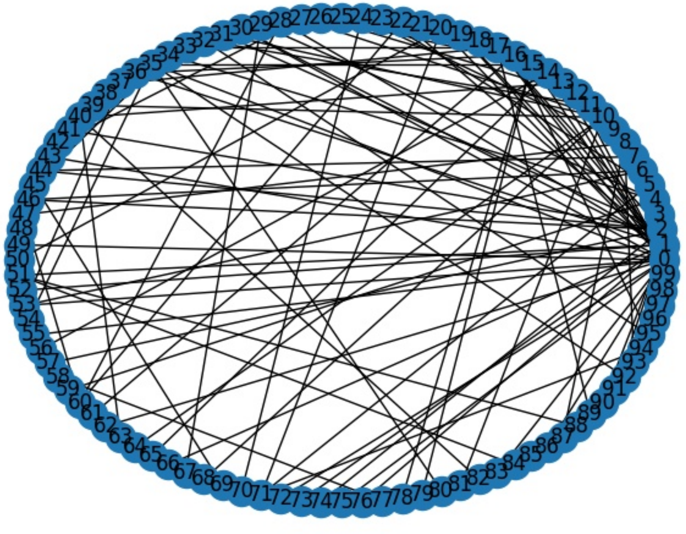


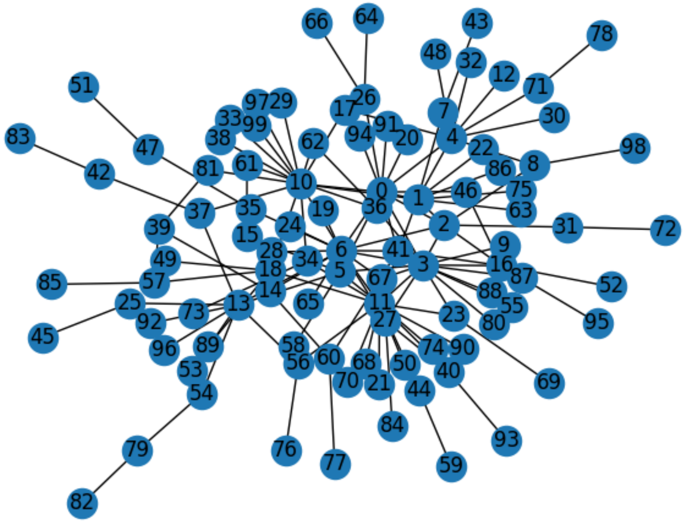




Scale-Free Networks

- Construction with preferential to given nodes (hubs)





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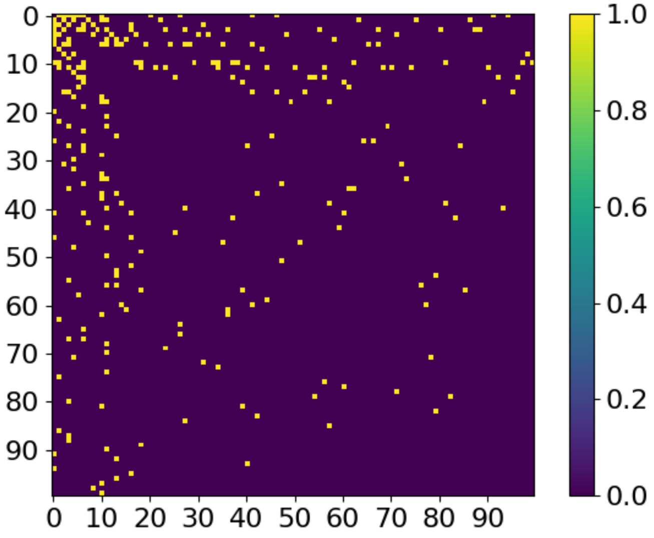
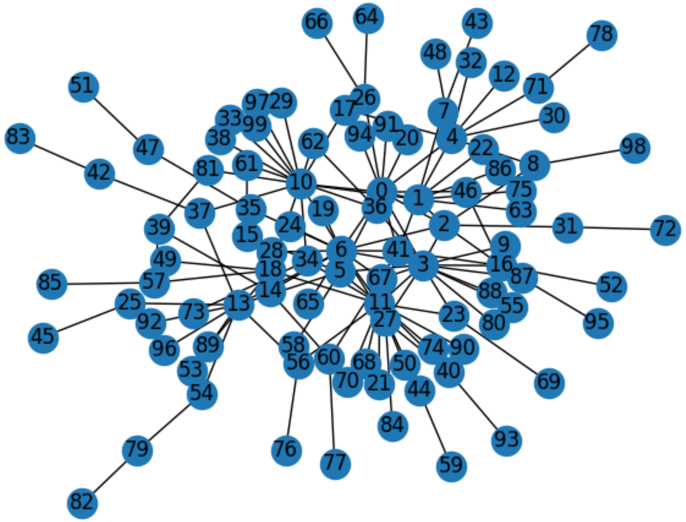
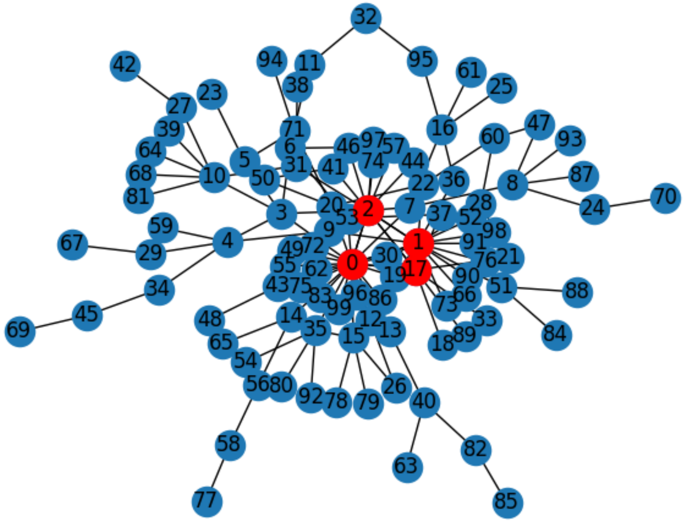




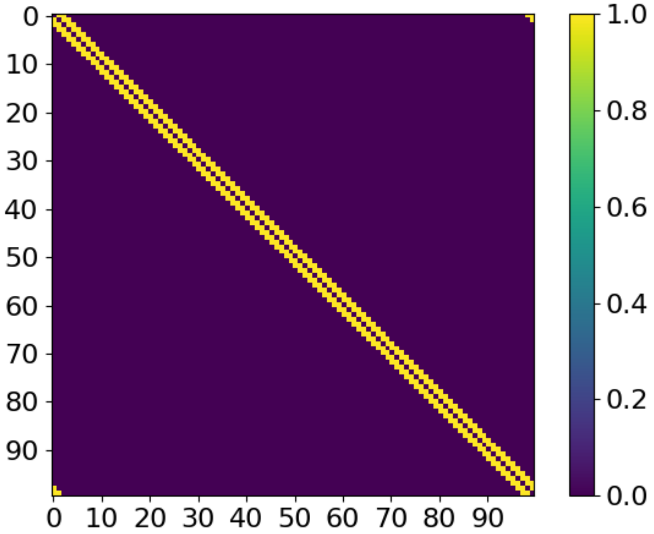
Scale-Free Networks

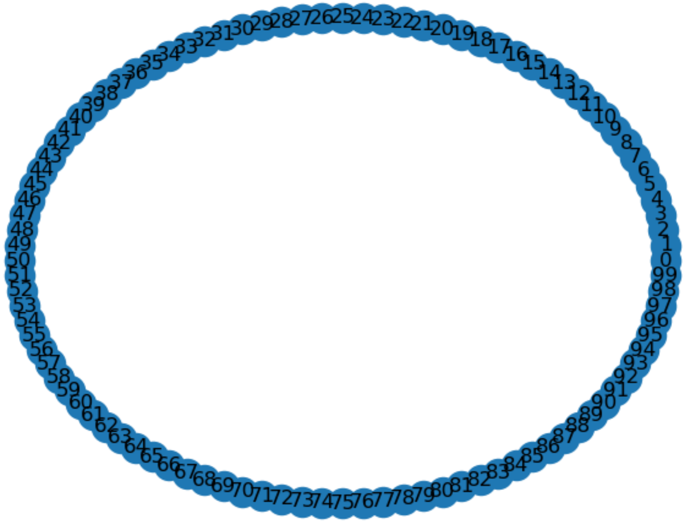
- Construction with preferential to given nodes (hubs)
- Core: subgraph with high degree hubs (here in red)
- Rich club: hubs that are connected together



Small-World Networks

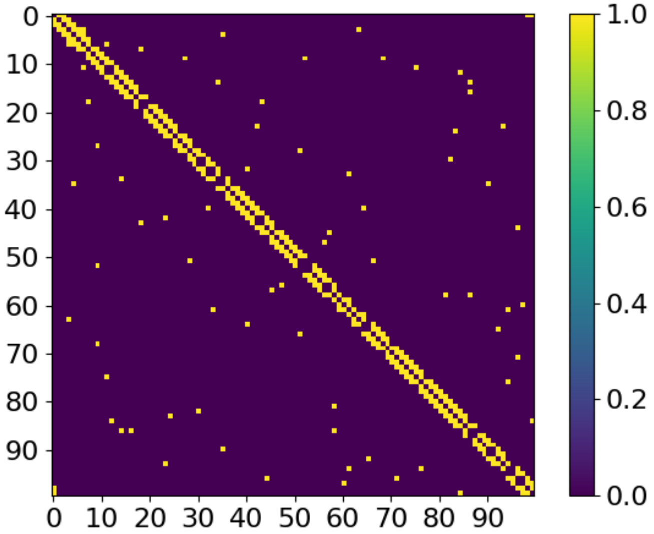
- Ring lattice with rewiring

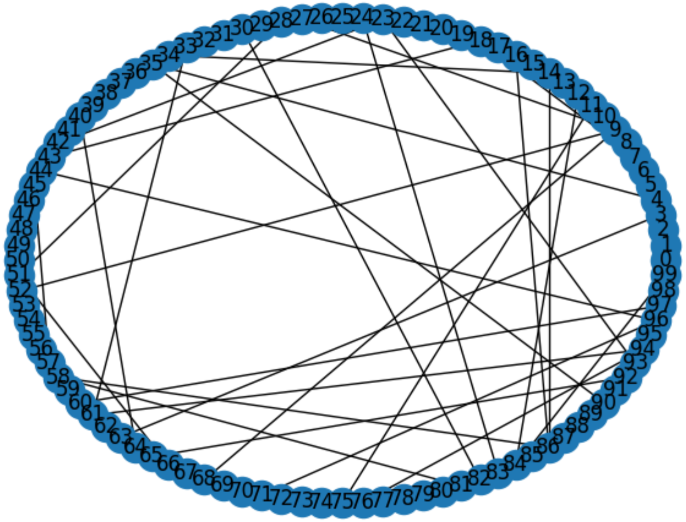




Small-World Networks

- Ring lattice with rewiring
- Rewired links are shortcuts to travel the graph
- No single important node, but distributed architecture with “easy traveling”





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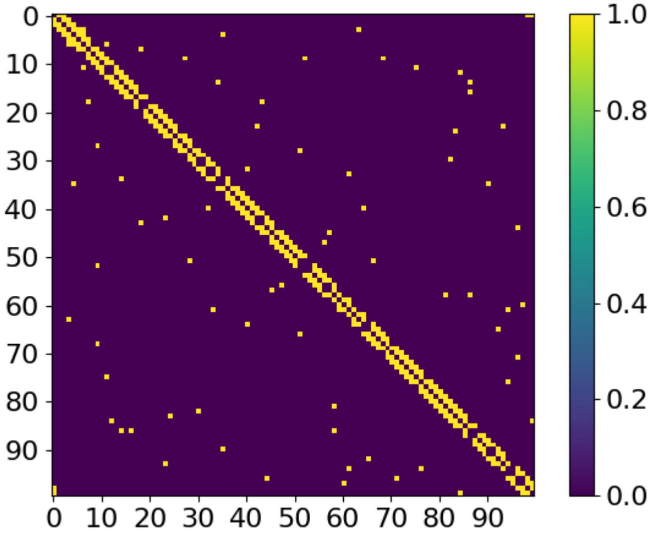

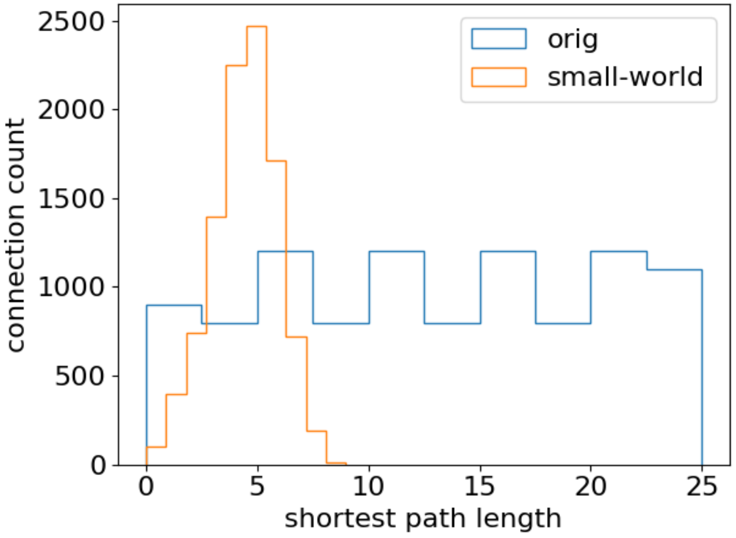

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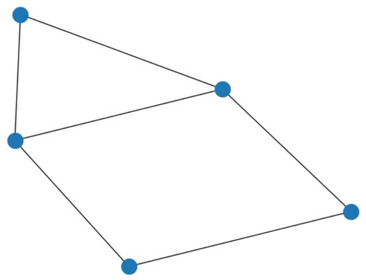
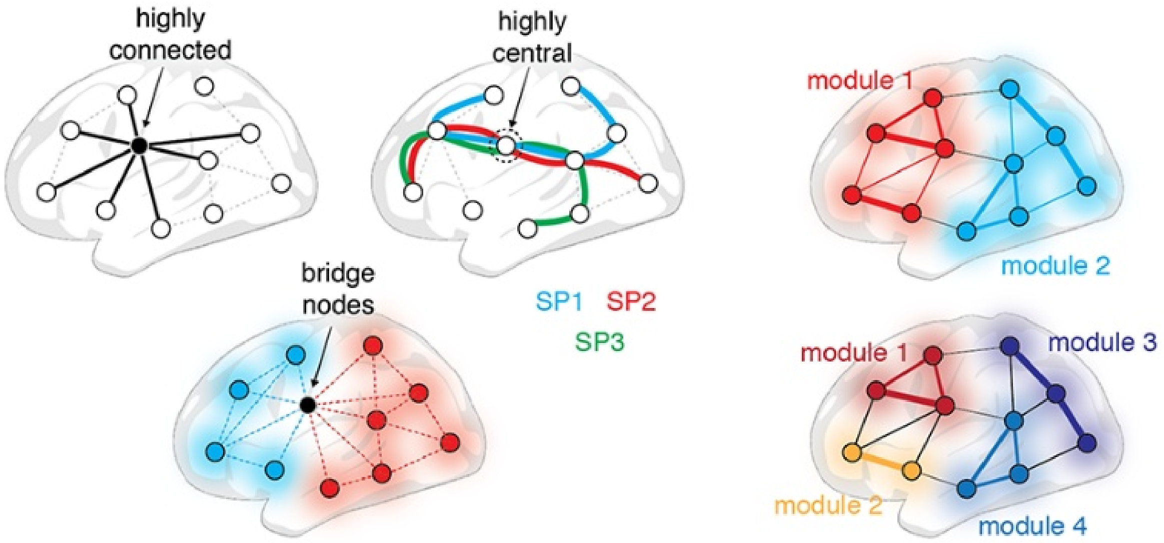
Directed and weighted graphs

- Node degree replaced by node strength $\sum_j A_{ij}$
- Some concepts and graph metrics can be easily transposed, others can't…
  - yes: hubs, community detection based on modularity, ...
  - no: shortest path (is the weight of a link a cost or an efficacy…?)



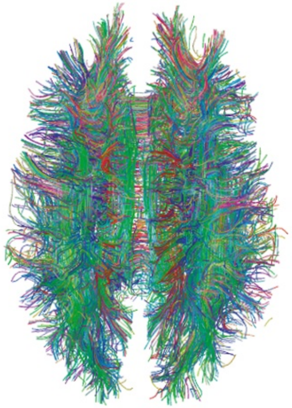
Network Analysis for MRI Data



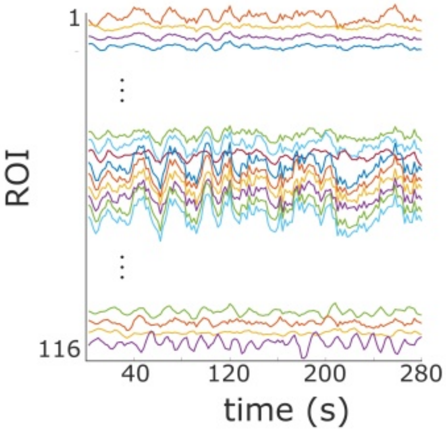
- **How to interpret network measures?**
- Structural connectivity (SC): anatomical white-matter fibers (e.g. density)
- Functional connectivity (FC): correlations of fMRI signals reflecting neuronal activity

Community Detection based on Structural versus Functional Data?

tractography (anatomy)



regional activity



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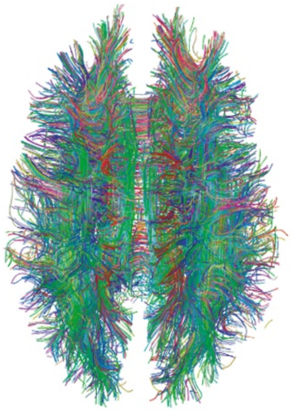

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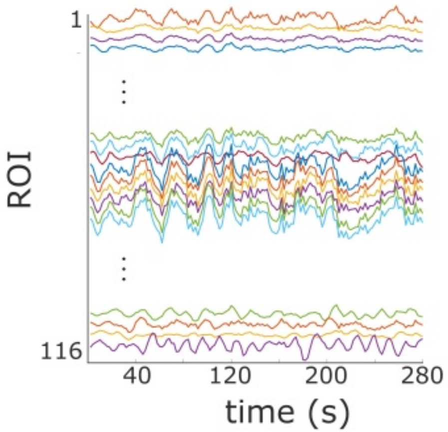

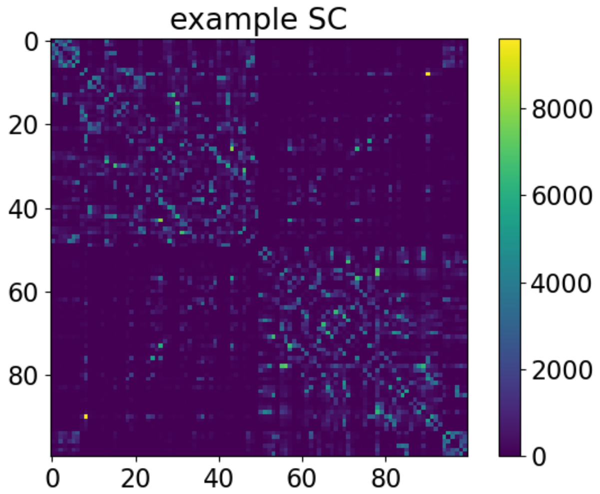

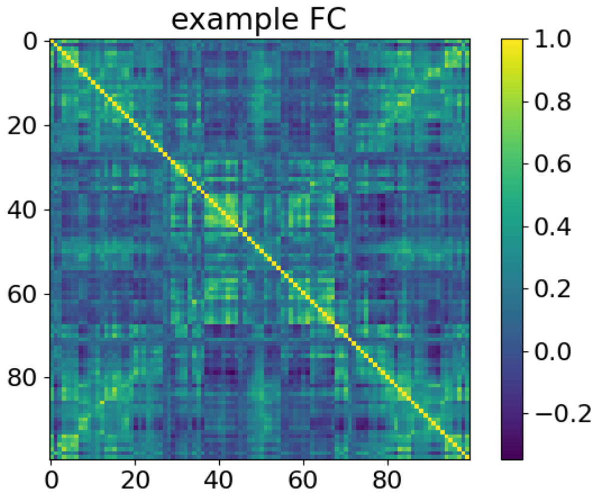

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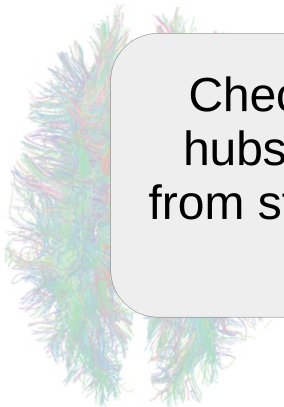

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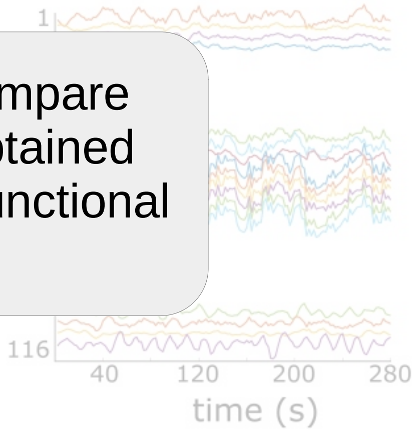
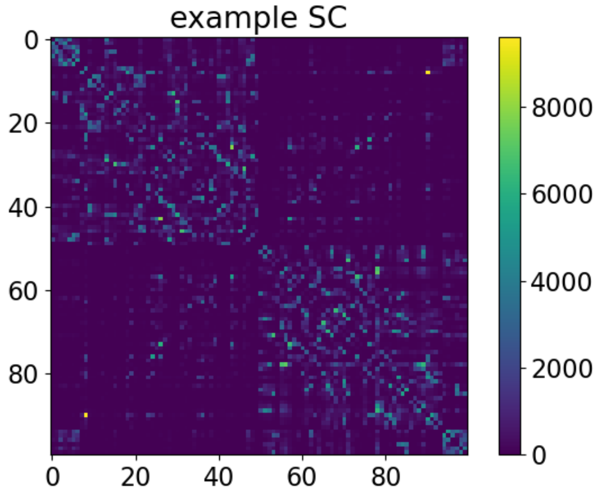
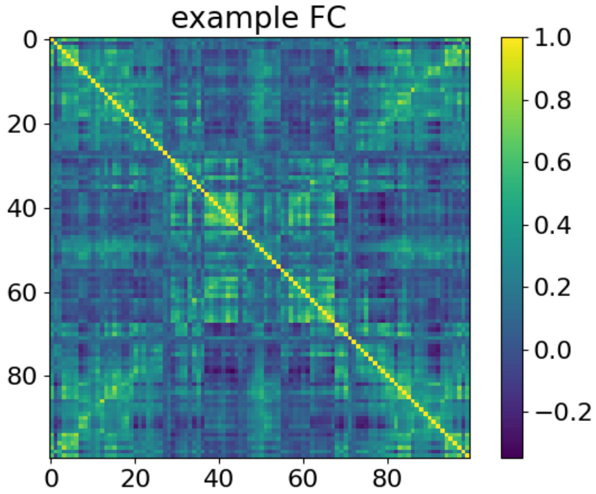


Check notebook to compare hubs / communities obtained from structural versus functional connectomes





Binarization of Matrices?

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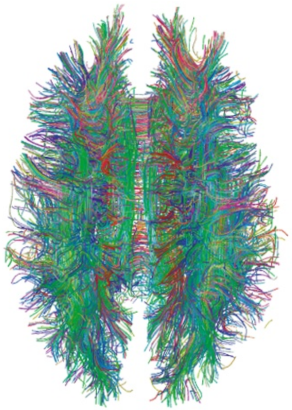
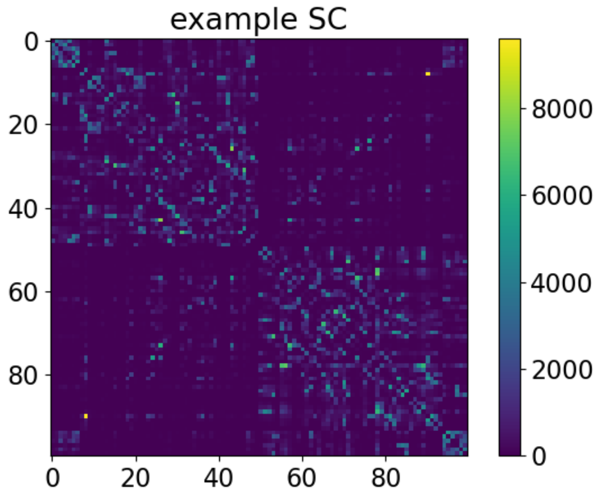
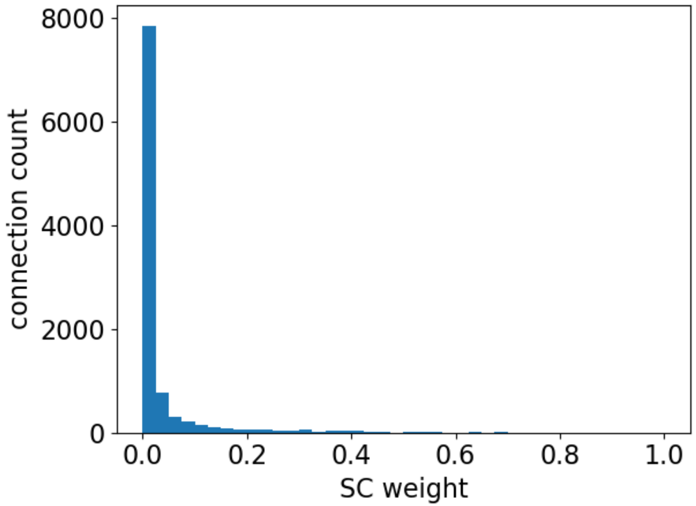




histogram of normalized weights



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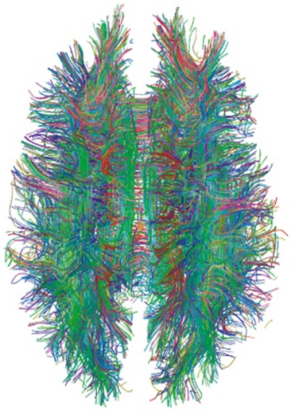

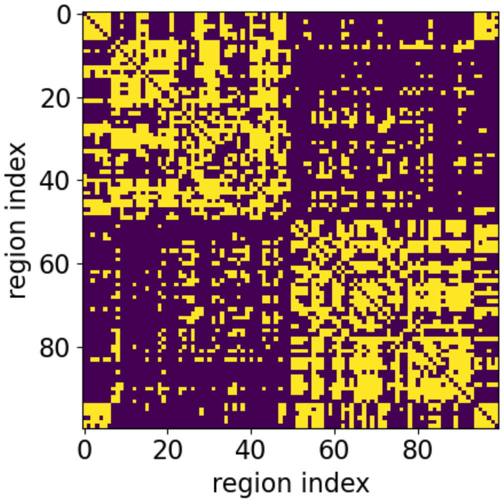

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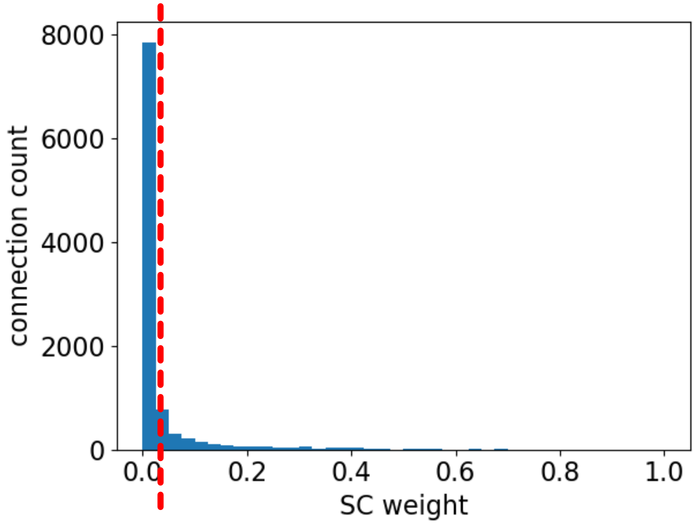

threshold?

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Practice

- Basic exercises in *basic_network_analysis*
  - undirected binary graphs
  - also first steps toward directed / weighted graphs
- Try real data: *ana_SC_FC.ipynb*
  - *ex_SC_HCP.npy*, *ex_SC.npy*, *ex_FC.npy*
  - identify hubs, core, communities